Roll No.							

24-MA-22

M.Sc. II SEMESTER [MAIN/ATKT] EXAMINATION
JUNE - JULY 2024

MATHEMATICS
Paper - II
[Lebesgue Measure and Integration]

[Max. Marks : 75] *[Time : 3:00 Hrs.]* *[Min. Marks : 26]*

Note : Candidate should write his/her Roll Number at the prescribed space on the question paper.
Student should not write anything on question paper.
Attempt five questions. Each question carries an internal choice.
Each question carries **15 marks.**

Q. 1 a) Let $\{A_n\}$ be a countable collection of sets of real numbers. Then prove that

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$$

b) If E_1 and E_2 are any measurable sets then prove that
$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

OR

a) Show that a continuous function defined on a measurable set is measurable.

b) if f is a measurable function defined on a measurable set E and if f and g are equivalent (equal a.e.) functions, then prove that g is a measurable function on E.

Q. 2 a) Let f be a bounded function defined on [a, b]. If f is Riemann integrable on [a, b] then show that it is Lebesgue integrable on [a, b] and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$

b) Let f be L - integrable i.e. bounded measurable function defined on a set E of finite measure and if $\alpha \leq f(x) \leq \beta$, then prove that ;
$\alpha\, m(E) \leq \int_E f(x)\,dx \leq \beta\, m(E)$.

OR

a) Let $\{f_n\}$ be a sequence of non - negative measurable functions and $f_n \to f$ a. e. on E. Then prove that $\int_E f \leq \underline{\lim}_{n \to \infty} \int_E f_n$

b) Let f be a bounded measurable function defined over a measurable set E. Then prove that $\left|\int_E f\right| \leq \int_E |f|$

Q. 3 a) Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} x \sin(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then prove that $D^{+} f(0) = D^{-} f(0) = 1$ and

$D_{+} f(0) = D_{-} f(0) = -1$

consequently, f is not differentiable at $x = 0$

b) Show that a function f is of bounded variation on [a, b] if and only if f is the difference of two monotone real valued functions on [a, b]

OR

a) Let $f \in [a, b]$ i.e. f be a Lebesgue integrable function on [a, b] then prove that indenfinite integral of f is a continuous function of bounded variation on [a, b].

b) Let f and g be two functions of bounded variation on [a, b] then prove that the function f + g is of bounded variation on [a, b].

Q. 4 a) Let $f \in L^{P}[a, b]$, $g \in L^{P}[a, b]$, then prove that $f + g \in L^{P}[a, b]$.

b) If f and g are square integrable in the Lebesgue sense, then prove that f + g is also square integrable in the Lebesgue sense and

$\| f + g \|_2 \leq \| f \|_2 + \| g \|_2$

OR

Let $1 \leq P < \infty$ and let $f, g \in L^{P}(\mu)$. Then prove that $f + g \in L^{P}(\mu)$ and

$\| f + g \|_P \leq \| f \|_P + \| g \|_P$

Q. 5 a) Let $\{f_n\}$ and $\{g_n\}$ be two sequences of measurable functions and let $f, g \in \mathcal{M}$

If $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ then prove that $f_n + g_n \xrightarrow{\mu} f + g$

b) Test the consistency of the following inequality

$$\int^{\pi} (f(x) - \sin x)^2 \, dx \leq \frac{4}{9} \quad \text{and} \quad \int^{\pi} (f(x) - \cos x)^2 \, dx \leq \frac{1}{9}$$

where $f \in L^{P}(0, \pi)$.

OR

Let $\{f_n\}$ be a sequence of measurable functions which converge to f a.e. on a measurable set E with $m(E) < \infty$. Then prove that for given $\eta > 0$ there is a set $A \subset E$ with $m(A) < \eta$ such that the sequence $\{f_n\}$ converges to f uniformly on $E - A$.

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